DETERMINANTS

Math 121, 11/14/2005 O.Knill

Determinants are characterized as the only n-linear alternating and normalized function from $M_n(F)$ to F.

DEFINITION. The **determinant** of a $n \times n$ matrix M over the field F is defined recursively by the Laplace (= cofactor) expansion

$$\boxed{\det(M) = \sum_{j=1}^n (-1)^{1+j} M_{1j} \det(\tilde{M}_{1j}}$$

where $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from M by deleting i'th row and j'th column.

EXAMPLE. The determinant of the 3×3 matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ in the field F_3 of characteristic 3 is

$$1\det(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}) - 2\det(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}) + 2\det(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}) = 1 \cdot 3 - 2 \cdot 1 + 2 \cdot (-1) = -1 = 2 \, .$$

REMARK. The usefulnes to consider also other fields F besides R or C had been demonstrated again recently in cryptology: on November 8'th, the number RSA-640 had been factored using an algorithm, which uses matrices defined over F_2.

TYPE 1 ELEMENTARY MATRICES have determinant -1. Example: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which switches the first and second row.

TYPE 2 ELEMENTARY MATRICES have determinant a. Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which multplies the second row with a.

TYPE 3 ELEMENTARY MATRICES have determinant 1. Example: $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which adds b times the third row to the first row.

PERMUTATION MATRIX. A **permutation** $\pi = \{\pi(1), ..., \pi(n)\}$ of $\{1, ..., n\}$ defines a permutation matrix $P_\pi e_j = e_{\pi(j)}$. Define the function $\sigma(\pi) = \det(P_\pi)$. on the group S_n of all permutations. If a permutation matrix M can be written as a product of k type 1 elementary matrices, then $\det(M) = (-1)^k$. In other words, $\sigma(PQ) = \sigma(P)\sigma(Q)$ for permutation matrices.

THEOREM. $\det(M) = \sum_{\pi \in S_n} \sigma(\pi) M_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}$

PROOF. Induction in n: group all permutations according to which element k is first. Each group is the set of all permutations of $n-1$ elements. So $\pi = [k, \rho(2), ..., \rho(n)]$ where ρ is a permutation of $\{1, ..., n\} \setminus \{k\}$. The signatures satisfy $\sigma(\pi) = (-1)^{k+1}\sigma(\rho)$. Grouping the terms in the sum appearing in the statement of the theorem gives the Laplace expansion.

REMARK. Without the $\sigma(\pi)$ part, the sum is called the **permanant** of M. It is not known whether there is a polynomial algorithm to computer permanents.

COROLLARY. We can also do Laplace expansion with respect to first column: $\det(A) = \sum_{i=1}^n (-1)^{i+1} \tilde{A}_{i1}$ or any column $\det(A) = \sum_{i=1}^n (-1)^{i+j} \tilde{A}_{ij}$ or any row $\det(A) = \sum_{j=1}^n (-1)^{i+j} \tilde{A}_{ij}$.

COROLLARY: $\det(A^T) = \det(A)$

Proof. For any permutation π define a **inverse permutation** by π^{-1} by a $P_\pi^T = P_\pi^{-1}$. We have $\sigma(\pi) = \sigma(\pi^{-1})$ because the inverse permutation has the same signature: interchanging two elements in the permutation changes sign. If one needs l such transpositions to unscramble the permuation then $\sigma(\pi) = (-1)^l$. But then, one needs also l transpositions to scramble it. Replacing each π with π^{-1} in the permutation formula does not change the value of the formula.

PROPERTIES. The determinant is a **normalized n-form**:

N-LINEARITY: The determinant is linear in a column (row), if the other columns (rows) fixed.
ALTERNATING: The determinant changes sign, if two columns (rows) are interchanged.
NORMALIZED: The determinant of the identity matrix is 1.

Proof: induction with repect to n. The statements for rows become statements for columns for the transpose matrix M^T so that it suffices to prove the statment for columns.
For $n = 2$, we have $\det(A) = A_{11}A_{22} - A_{12}A_{21}$ and all properties are true. Assume the property is true for $n-1$. To prove (i), distinguish the case, where the column is the first column and the case, where we have an other column. In the first case, we see the linearity directly, in the second case, we use induction. (ii) follows from the permutation formula. The third statement is obvious.

ROW REDUCTION. The determinant changes sign, if two rows are interchanged. The determinant is multplied by a factor a if a row is multiplied with a. The determinant does not change if a multiple of one row is added to an other row. Therefore, $\det(M) = (-1)^p \prod_{j=1}^q a_j$ where p is the number of row transpositions and a_j are the scaling factors applied during the row reductions.
REMARK: Row reduction is the most convenient algorithm to compute determinants. Instead of adding $n!$ terms, the row reduction algorithm only needs $O(n^3)$ steps since there are $O(n^2)$ row reduction steps each involving $O(n)$ operations.

COROLLARY. Any invertible matrix M is the product $M = E_1 \cdot ... \cdot E_k$ of elementary matrices.
Proof. This is a reformulation of what happens during row reduction.

COROLLARY. $\det(AB) = \det(A) \cdot \det(B)$.

Proof. If A is not invertible, its rank is smaller than n and so the rank of AB so that AB is not invertible and both sides of the formula are the same. In the invertible case, $\det(M) = \prod_i \det(E_i)$. Because the multiplicative formula is true if the first matrix is an elementary the formula is true in general by inducuction.

COROLLARY. $\det(A) \neq 0$ if and only if A is invertible.
Proof. If A is invertible, then $AB = 1$ for some matrix B. Because det is multiplicative $\det(A)\det(B) = 1$ so that $\det(A) \neq 0$. If $\det(A) \neq 0$, then each of the elementary matrices building up A is invertible and so also A.

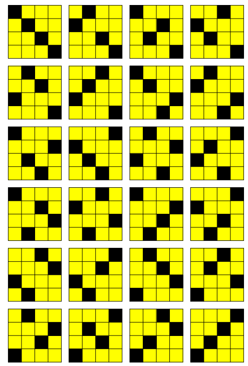
THEOREM. If $\delta : M_n(F) \to F$ is n-linear, alternating and normalized, then $\delta = \det$.

PROOF. step (i): If E_1 is an elementary matrix of type 1, then $\delta(E_1) = (-1)$
if E_2 is an elementary matrix of type 2, then $\delta(E_2) = a$
If E_3 is an elementary matrix of type 3, then $\delta(E_3) = 1$.
step (ii): it follows that δ is multiplicative $\delta(AB) = \delta(A)\delta(B)$: Proof: check this first if A is an elementary matrix. Because every inverible matrix is a product of elementary matrices, it is true in general.
step (iii): if A is invertible, then $\delta(A) \neq 0$. Proof: there exists then a matrix B such that $AB = 1$. Because of the multiplicative property (ii) and the assumption $\delta(1) = 1$, it can not be that $\delta(A) = 0$.
step (iv): it follows that if $M = E_1 \cdot ... \cdot E_k$ is the product of k elementary matrices, then $\det(M) = \prod_i \det(E_i)$.
step (v): it follows that if M is invertible, then each E_i is invertible and since for invertible elementary matrices $\delta(E) \neq 0$, also $\delta(M) \neq 0$.
step (vi): if A is not invertible, then $\delta(A) = 0$, the same as $\det(A)$. step (vi): for invertible matrices, we can write $M = \prod_{i=1}^r E_i$. Because $\delta(E_i) = \det(E_i)$, we get delta$(M) = \delta(\prod_{i=1}^r E_i) = \prod_{i=1}^r \delta(E_i) = \prod_{i=1}^r \det(E_i) = \delta(\prod_{i=1}^r E_i) = \delta(M)$.

DETERMINANTS II — **Math 121, 11/16/2005 O.Knill**

A summary of all properties about determinants.

PERMUTATIONS. A **permutation** of $\{1,2,\dots,n\}$ is a rearrangement of $\{1,2,\dots,n\}$. There are $n! = n\cdot(n-1)...\cdot 1$ different permutations of $\{1,2,\dots,n\}$: fixing the position of first element leaves $(n-1)!$ possibilities to permute the rest.



EXAMPLE. There are 6 permutations of $\{1,2,3\}$: $(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)$.

PATTERNS AND SIGN. The matrix A with zeros everywhere except $A_{i,\pi(i)}=1$ is called a **permutation matrix** or the **pattern** of π. An **inversion** is a pair $k<l$ such that $\sigma(k)<\sigma(l)$. The **sign** of a permutation π, denoted by $(-1)^{\sigma(\pi)}$ is (-1) for an odd number of inversions in the pattern, otherwise, the sign is 1. (To get the sign in the permutations to the right, count the number of pairs of black squares, where the upper square is to the right).

EXAMPLES. $\sigma(1,2)=0$, $\sigma(2,1)=1$. $\sigma(1,2,3)=\sigma(3,2,1)=\sigma(2,3,1)=1$. $\sigma(1,3,2)=\sigma(3,2,1)=\sigma(2,1,3)=-1$.

DETERMINANT The **determinant** of a $n\times n$ matrix A is defined as the sum $\boxed{\sum_\pi \sigma(\pi)A_{1\pi(1)}A_{2\pi(2)}\cdots A_{n\pi(n)}}$, where π is a permutation of $\{1,2,\dots,n\}$ and $\sigma(\pi)$ is its sign.

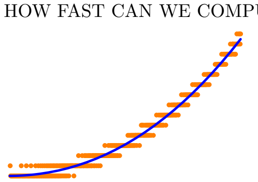
2×2 CASE. The determinant of $A=\begin{vmatrix} a & b \\ c & d\end{vmatrix}$ is $ad-bc$. There are two permutations of $(1,2)$. The identity permutation $(1,2)$ gives $A_{11}A_{12}$, the permutation $(2,1)$ gives $A_{21}A_{22}$.
If you have seen some multi-variable calculus, you know that $\det(A)$ is the area of the parallelogram spanned by the column vectors of A. The two vectors form a basis if and only if $\det(A)\neq 0$.

3×3 CASE. The determinant of $A=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i\end{vmatrix}$ is $aei+bfg+cdh-ceg-fha-bdi$ corresponding to the 6 permutations of $(1,2,3)$. Geometrically, $\det(A)$ is the volume of the parallelepiped spanned by the column vectors of A. The three vectors form a basis if and only if $\det(A)\neq 0$.

EXAMPLE DIAGONAL AND TRIANGULAR MATRICES. The determinant of a diagonal or triangular matrix is the product of the diagonal elements.

EXAMPLE PERMUTATION MATRICES. The determinant of a matrix which has everywhere zeros except $A_{i\pi(j)}=1$ is just the sign $\sigma(\pi)$ of the permutation.

HOW FAST CAN WE COMPUTE THE DETERMINANT?.



The cost to find the determinant is the same as for the Gauss-Jordan elimination as we will see below The graph to the left shows some measurements of the time needed for a CAS to calculate the determinant in dependence on the size of the $n\times n$ matrix. The matrix size ranges from n=1 to n=300. We also see a best cubic fit of these data using the least square method from the last lesson. It is the cubic $p(x)=a+bx+cx^2+dx^3$ which fits best through the 300 data points.

WHY DO WE CARE ABOUT DETERMINANTS?

- check invertibility of matrices
- have geometric interpretation as volume
- explicit algebraic expressions for inverting a matrix
- as a natural functional on matrices it appears in formulas in particle or statistical physics
- allow to define orientation in any dimensions
- appear in change of variable formulas in higher dimensional integration.
- proposed alternative concepts are unnatural, hard to teach and and harder to understand
- determinants are fun

TRIANGULAR AND DIAGONAL MATRICES.
The determinant of a **diagonal** or **triangular** matrix is the product of its diagonal elements.

Example: $\det(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1\end{bmatrix}) = 20$.

PARTITIONED MATRICES.
The determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B\end{bmatrix}$ is the product $\det(A)\det(B)$.

Example $\det(\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2\end{bmatrix}) = 2\cdot 12 = 24$.

LINEARITY OF THE DETERMINANT. If the columns of A and B are the same except for the i'th column,

$$\boxed{\det([v_1,...,v,...v_n]) + \det([v_1,...,w,...v_n]) = \det([v_1,...,v+w,...v_n])}$$

In general, one has $\boxed{\det([v_1,...,kv,...v_n]) = k\det([v_1,...,v,...v_n])}$ The same identities hold for rows and follow directly from the original definition of the determimant.

PROPERTIES OF DETERMINANTS.

$\boxed{\det(AB)=\det(A)\det(B)}$ $\boxed{\det(SAS^{-1})=\det(A)}$ $\boxed{\det(\lambda A)=\lambda^n\det(A)}$

$\boxed{\det(A^{-1})=\det(A)^{-1}}$ $\boxed{\det(A^T)=\det(A)}$ $\boxed{\det(-A)=(-1)^n\det(A)}$

If B is obtained from A by switching two rows, then $\det(B)=-\det(A)$. If B is obtained by adding an other row to a given row, then this does not change the value of the determinant.

PROOF OF $\det(AB)=\det(A)\det(B)$, one brings the $n\times n$ matrix $[A|AB]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB]=[1|B]$. By looking at the $n\times n$ matrix to the left during Gauss-Jordan elimination, the determinant has changed by a factor $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB)=\det(A)\det(B)$.
PROOF OF $\det(A^T)=\det(A)$. The transpose of a pattern is a pattern with the same signature.

PROBLEM. Find the determinant of $A=\begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4\end{bmatrix}$.

SOLUTION. Three row transpositions give $B=\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 2\end{bmatrix}$ a matrix which has determinant 84. Therefore $\det(A)=(-1)^3\det(B)=-84$.

PROBLEM. Determine $\det(A^{100})$, where A is the matrix $\begin{vmatrix} 1 & 2 \\ 3 & 16\end{vmatrix}$.

SOLUTION. $\det(A)=10$, $\det(A^{100})=(\det(A))^{100}=10^{100}=1\cdot$gogool. This name as well as the gogoolplex $=10^{10^{100}}$ are official. They are huge numbers: the mass of the universe for example is $10^{52}kg$ and $1/10^{10^{51}}$ is the chance to find yourself on Mars by quantum fluctuations. (R.E. Crandall, Scient. Amer., Feb. 1997).

ROW REDUCED ECHELON FORM. Determining rref(A) also determines $\det(A)$.
If A is a matrix and α_i are the factors which are used to scale different rows and s is the number of times, two rows are switched, then $\boxed{\det(A)=(-1)^s\alpha_1\cdots\alpha_n\det(\text{rref}(A))}$.

INVERTIBILITY. Because of the last formula: $\boxed{\text{A } n\times n \text{ matrix } A \text{ is invertible if and only if } \det(A)\neq 0.}$

GENERAL LAPLACE EXPANSION. We compute the determinant of $n \times n$ matrices $A = a_{ij}$. Choose a column i. For each entry a_{ji} in that column, take the $(n-1) \times (n-1)$ matrix A_{ij} called **minor** which does not contain the i'th column and j'th row. One gets

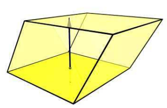
$$\boxed{\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})}$$

This Laplace expansion just arranges the permutations: listing all permutations of the form $(1, *, ..., *)$ of n elements is the same then listing all permutations of $(2, *, ..., *)$ of $(n-1)$ elements.

ORTHOGONAL MATRICES. Because $Q^T Q = 1$, we have $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$ Rotations have determinant 1, reflections have determinant -1.

QR DECOMPOSITION. If $A = QR$, then $\det(A) = \det(Q)\det(R)$. The determinant of Q is ± 1 times the determinant of R is the product of the diagonal elements of R.

DETERMINANT AND VOLUME. If A is a $n \times n$ matrix and the field F is the real line, then $|\det(A)|$ is the volume of the n-dimensional parallelepiped E_n spanned by the n column vectors v_j of A.

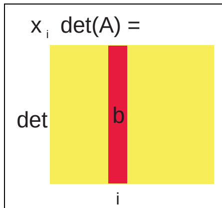


Proof. Use the QR decomposition $A = QR$, where Q is orthogonal and R is upper triangular. From $QQ^T = 1$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \pm\det(R)$. The determinant of R is the product of the $||u_i|| = ||v_i - \text{proj}_{V_{j-1}} v_i||$ which was the distance from v_i to V_{j-1}. The volume $\text{vol}(E_j)$ of a j-dimensional parallelepiped E_j with base E_{j-1} in V_{j-1} and height $||u_i||$ is $\text{vol}(E_{j-1})||u_j||$. Inductively $\text{vol}(E_j) = ||u_j|| \text{vol}(E_{j-1})$ and therefore $\text{vol}(E_n) = \prod_{j=1}^{n} ||u_j|| = \det(R)$.

$$\boxed{\text{The volume of a } k \text{ dimensional parallelepiped defined by the vectors } v_1, \dots, v_k \text{ in } R^n \text{ is } \sqrt{\det(A^T A)}.}$$

Proof. $Q^T Q = I_n$ gives $A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R$. So, $\det(R^T R) = \det(R)^2 = (\prod_{j=1}^{k} ||u_j||)^2$. (Note that A is a $n \times k$ matrix and that $A^T A = R^T R$ and R are $k \times k$ matrices.)

ORIENTATION. Determinants allow to **define** the orientation of n vectors in n-dimensional space. They are the "right hand rule" in hyperspace... To do so, define the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.



CRAMER'S RULE. This is an explicit formula for the solution of $A\vec{x} = \vec{b}$. If A_i denotes the matrix, where the column $\vec{v}_i$ of A is replaced by $\vec{b}$, then

$$\boxed{x_i = \det(A_i)/\det(A)}$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n] = \det([v_1, \dots, (Ax), \dots, v_n] = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n] = x_i \det([v_1, \dots, v_i, \dots, v_n]) = x_i \det(A)$

EXAMPLE. Solve the system $5x+3y = 8, 8x+5y = 2$ using Cramers rule. This linear system with $A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. We get $x = \det \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix} = 34 y = \det \begin{bmatrix} 5 & 8 \\ 8 & 2 \end{bmatrix} = -54$.

GABRIEL CRAMER. (1704-1752), born in Geneva, Switzerland, he worked on geometry and analysis. He died during a trip to France, where he wanted to start his retirement. Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébraique", where he solved like this a system of equations with 5 unknowns. According to a short biography of Cramer by J.J O'Connor and E.F. Robertson, the rule had however been used already before by other mathematicians. Solving systems with Cramers formulas is slower than by Gaussian elimination. The rule is still important. For example, if A or b depends on a parameter t, and we want to see how x depends on the parameter t one can find explicit formulas for $(d/dt)x_i(t)$. Cramers rule tells for example that the solution can depend in a sensitive way on parameters if the determinant is small.

THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $A\vec{x} = \vec{e}_i$, where $\vec{e}_j$ are basis vectors, Cramers rule together with the Laplace expansion gives the formula:

$$\boxed{[A^{-1}]_{ij} = (-1)^{i+j} \det(A_{ji})/\det(A)}$$

$B_{ij} = (-1)^{i+j} \det(A_{ji})$ is called the **classical adjoint** of A. **Note** the change $ij \to ji$. **Don't** confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.

EXAMPLE. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \\ 6 & 0 & 7 \end{bmatrix}$ has $\det(A) = -17$ and we get $A^{-1} = \begin{bmatrix} 14 & -21 & 10 \\ -11 & 8 & -3 \\ -12 & 18 & -11 \end{bmatrix} /(-17)$:

$B_{11} = (-1)^{1+1} \det \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} = 14$. $B_{12} = (-1)^{1+2} \det \begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix} = -21$. $B_{13} = (-1)^{1+3} \det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 10$.

$B_{21} = (-1)^{2+1} \det \begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} = -11$. $B_{22} = (-1)^{2+2} \det \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} = 8$. $B_{23} = (-1)^{2+3} \det \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = -3$.

$B_{31} = (-1)^{3+1} \det \begin{bmatrix} 5 & 2 \\ 6 & 0 \end{bmatrix} = -12$. $B_{32} = (-1)^{3+2} \det \begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix} = 18$. $B_{33} = (-1)^{3+3} \det \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} = -11$.

THE ART OF CALCULATING DETERMINANTS. When confronted with a matrix, it is good to go through a checklist of methods to crack the determinant. Often, there are different possibilities to solve the problem, in many cases the solution is particularly simple using one method.

- Is it a 2×2 or 3×3 matrix?
- Do you see duplicated columns or rows?
- Is it a upper or lower triangular matrix?
- Can you row reduce to a triangular case?
- Is it a partitioned matrix?
- Are there only a few nonzero patters?
- Is it a trick: like A^{1000}?
- Laplace expansion with some row or column?
- Does geometry imply noninvertibility?
- Later: Can you see the eigenvalues of $A - \lambda I_n$?

EXAMPLES.

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$ Try row reduction.

2) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ Laplace expansion.

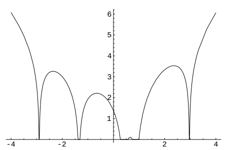
3) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$ Partitioned matrix.

4) $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Make it triangular.

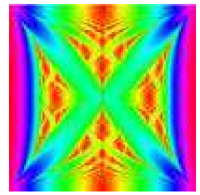
APPLICATION HOFSTADTER BUTTERFLY. In solid state physics, one is interested in the function $f(E) = \det(L - EI_n)$, where

$$L = \begin{bmatrix} \lambda\cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda\cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda\cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda\cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the **spectrum** of the quantum mechanical system. A physicist is interested in the rate of change of $f(E)$ or its dependence on λ when E is fixed. .



The graph to the left shows the function $E \mapsto \log(|\det(L - EI_n)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture shows the spectrum of the crystal depending on α. It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher Bach" by Douglas Hofstadter.



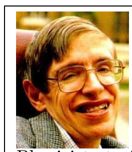
CHANGE OF VARIABLES. If $x \mapsto y = u(x)$ is a change of variable, then the Jacobean matrix $Du(x)$ is the linearization of the map near x and $||dy|| = |\det(Du(x))| \cdot ||dx||$. This leads to the change of variable formula

$$\int_S f(x)\, dx = \int_{u(S)} f(y)\, |\det(Du^{-1}(y)| dy .$$

If u is a map from $\mathbf{R}^m$ to $\mathbf{R}^n$, where $m \leq n$, then the expansion factor is $\sqrt{\det(A^T A)}$, where $A = Du$.

Example: (1 dim) if $x = u^{-1}(y) = \sin(y)$, $dx = \cos(y)\, dy$. $\int_0^1 \sqrt{1-x^2}\, dx = \int_0^{\pi/2} \sqrt{(1-\sin^2(y))}\, \cos(y)\, dy = \int_0^{\pi/2} \cos^2(y)\, dy = \pi/4$.
Example (2 dim) If $u(s,t) = (x(s,t), y(s,t), x(s,t))$ is a surface, then $A = Du(s,t)$ is a 3×2 matrix with column vectors $X = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix}$ $Y = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix}$ (which are tangent vectors to the surface). Now $A^T A = \begin{bmatrix} X \cdot X & X \cdot Y \\ X \cdot Y & Y \cdot Y \end{bmatrix}$ whose determinant is $||X||^2||Y||^2 - ||X \cdot Y||^2 = ||X||^2||Y||^2(1-\cos(\phi)^2) = ||X||^2||Y||^2\sin(\phi)^2 = ||X \times Y||^2$. The expansion factor is $\sqrt{\det(A^T A)} = ||X \times Y||$.

VAN DER MONDE DETERMINANT. If we define for a scalar a the vector $\vec{a} = \begin{bmatrix} 1, a, a^2, \dots, a^n \end{bmatrix}$. For $n+1$ scalars $a_0, \dots, a_n$ we consider the $(n+1)\times(n+1)$-matrix with rows $\vec{a_i}$.
CLAIM: $\det(A) = \prod_{i>j}(a_i - a_j)$. To prove this, we make a Laplace expansion with respect to the last column. To do so, we call $a_n = x$ and see that the determinant is a polynomial $f(x)$ of degree n in x. It satisfies $f(a_0) = f(a_1) = ... = f(a_{n-1}) = 0$ because in those cases, the determinant is zero. Therefore $f(x) = k(x-a_1)\dots,(x-a_{n-1})$ for some constant k. The number k is the coefficient in front of x^n: it is the van der Monde determinant in the case $n-1$. The recursion $\mathrm{VM}_n = \prod_{n>j}(a_n - a_j)\mathrm{VM}_{n-1}$ proves the claim.



DETERMINANTS IN PHYSICS. Physicists are excited about determinants because summation over all possible "paths" is used as a quantization method. The Feynmann path integral is a "summation" over a suitable class of paths and leads to quantum mechanics. The relation with determinants comes because each summand in a determinant can be interpreted as a contribution of a path in a finite graph with n nodes.
Physicists trying to glue quantum mechanics with general relativity hope to make sense of expressions like $\int_g e^{i\det(A(g))}$, where A is an operator attached to some geometry g and where the integral "sums" over all possible geometries.

A DETERMINANT OVER AN OTHER FIELD. Let F be the field of rational functions $p(x)/q(x)$ where p, q are polynomials over the reals and let

$$A = \begin{bmatrix} x^2 + 3x + 1 & 2x \\ -3x + 1 & 1 \end{bmatrix} .$$

Then $\det(A) = 7x^2 + x + 1$.

DETERMINANTS IN NUMBER THEORY. Does the system of equations

$$\begin{aligned} 3x + 5y &= a \bmod 11 \\ x + y &= b \bmod 11 \end{aligned}$$

have solutsions for all a, b.

Answer: this is a linear system of equations over the field F_{11}. The determinant is 9 in that field which is nonzero. So, there is always a solution.

REMARK. When working over a field F_p with prime p, we always can compute the determinant of the matrix as we are used to over R, then take the result modulo p.

JACOBIS DETERMINANT IDENTITY.

If $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} V & W \\ X & Y \end{bmatrix}$ then $\det(A)\det(Y) = \det(B)$.

PROOF. For triangular **partitioned matrices** the determinant is the product of the determinants of the diagonal matrices. Use now the identity

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix} \begin{bmatrix} 1 & W \\ 0 & Y \end{bmatrix} = \begin{bmatrix} B & 0 \\ D & 1 \end{bmatrix}$$